ISRAEL JOURNAL OF MATHEMATICS, Vol. 51, Nos. 1-2, 1985

CATEGORY THEOREMS FOR SOME ERGODIC MULTIPLIER PROPERTIES

BY
JON AARONSON
Department of Mathematics, Tel Aviv University, Ramat Aviv, Israel

ABSTRACT
We prove (Baire) category theorems for ergodic multiplier properties stronger than weak mixing, and weaker than mild mixing.

§1. Introduction

In this paper we "identify the Baire categories" of some types of ergodic multiplier properties lying between weak mixing and mild mixing.

Let $(X, \mathcal{B}, m)$ denote the unit interval equipped with Lebesgue measure and let $S : X \to X$ be a measure preserving transformation. Then (by the weak mixing theorem [5]) S is weakly mixing if and only if $S \times T$ is ergodic for every ergodic measure preserving transformation T of X. This means that weak mixing is an ergodic multiplier property in the sense which we proceed to define.

Denote by G, the group of non-singular invertible transformations of X, G^e the collection of ergodic ones and G_m the collection of invertible transformations of X preserving m. We think of an ergodic multiplier property (in G_m) as the property of belonging to an ergodic multiplier set (in G_m) which is a set of the form

$$E(P) = \{S \in G_m : S \times T \text{ is ergodic for every } T \in P\}$$

where $P \subseteq G$. Evidently if $E(P)$ is non-empty, $P \subseteq G^e$ and if $P \subseteq Q$ then $E(P) \supseteq E(Q)$. We see that indeed weak mixing is an ergodic multiplier property in this sense, the collection of weak mixing transformations in G_m being $E(G_m \cap G^e)$.

Received January 24, 1983 and in revised form September 7, 1984

P. Halmos showed (cf. [4] and [5]) that the collection of weakly mixing transformations in G_m is residual in G_m (*endowed with the weak topology*).

Recall from [3] that $S \in G_m$ is called rigid if there is a sequence $n_\kappa \to \infty$ such that $m(A \triangle S^{-n_\kappa}A) \to_{\kappa\to\infty} 0$ for every $A \in \mathcal{B}$, and that $S \in G_m$ is called mildly mixing if S has no non-trivial rigid factor. It is proved in [3] that the collection of mildly mixing transformations in G_m is $E(G^e)$.

It follows from results of Katok and Stepin that the collection of rigid transformations is residual in G_m. Thus $E(G^e)$, the collection of mildly mixing transformations, is meagre in G_m.

It is known (see §2) that if $S \in G_m \cap G^e$ and $T \in G^e$ then $S \times T$ is ergodic if and only if the eigenvalues of T are a null-set for the spectral measure of S.

Now $S \in G_m$ is weakly mixing if and only if σ_s (the spectral measure of S) is non-atomic, and so for S weakly mixing, $S \times T$ is ergodic for every $T \in G^e$ with a countable eigenvalue group.

Set $G^e(\aleph_0) = \{T \in G^e : T\text{'s eigenvalue group is countable}\}$. We see that by Halmos' result:

$$E(G^e(\aleph_0)) \textit{ is residual in } G_m.$$

In general, the eigenvalues of $T \in G^e$ form a Borel subset of the circle witl Lebesgue measure zero. However ([2]), for every $\rho(t) > 0$, $\rho(t) \downarrow 0$, $\rho(t)/t \uparrow \infty$ as $t \downarrow 0$, there is a $T \in G^e$ whose eigenvalues have positive ρ-Hausdorff measure. For ρ as above, let

$$G^e(\rho) = \{T \in G^e : T\text{'s eigenvalues have } \rho\text{-Hausdorff measure zero}\}.$$

Our first result (Theorem 1) is that for any such ρ, $E(G^e(\rho))$ *is meagre in* G_m. This has the interpretation that "*in general*" (see [5]) *a transformation has small spectrum* — its spectral measure charges small eigenvalue groups.

Let $c_n \downarrow 0$ as $n \uparrow \infty$, $\sum_{n=1}^\infty c_n = \infty$. Recall from [8] that $T \in G^e$ is c_n-recurrent if

$$\sum_{n=1}^{\infty} c_n f \cdot T^n \frac{dm \cdot T^n}{dm} = \infty \qquad \text{a.e.}$$

for every non-negative measurable function f with $\int_X f dm > 0$ (for a relationship between c_n-recurrence and size of eigenvalue groups, see [2]). For $c = (c_1, c_2, \ldots)$ where $c_n \downarrow 0$, $\sum_{n=1}^\infty c_n = \infty$; let $G^e(c) = \{T \in G^e : T \text{ is } c_n\text{-recurrent}\}$.

Theorem 2 says that for any such c (however small c_n), $E(G^e(c))$ *is meagre in* G_m.

We finish the paper with a class of residual ergodic multiplier properties.

Suppose $a(n) \to \infty$, $a(n)/n \to 0$ as $n \to \infty$. Let

$$G^e(\{a(n)\}) = \left\{ T \in G^e : \lim_{n\to\infty} \frac{1}{a(n)} \sum_{\kappa=0}^{n-1} m(A \cap T^{-\kappa}B) > 0 \text{ for every } A, B \in \mathscr{B}; \right.$$
$$\left. m(A), m(B) > 0 \right\}.$$

(It is not hard to show, using the Chacon–Ornstein theorem, that every $T \in G^e$ is in some $G^e(\{a(n)\})$, and also, that if $T \in G^e(\{a(n)\})$, $c_n \downarrow 0$ as $n \uparrow \infty$ and $\sum_{n=1}^{\infty}(c_n - c_{n+1})a(n) = \infty$ then T is c_n-recurrent.)

Theorem 3 states that for every $a(n) \to \infty$, $a(n)/n \to 0$ as $n \to \infty$,

$$E(G^e(\{a(n)\})) \textit{ is residual.}$$

The author would like to thank M. Keane for some helpful discussions.

§2. The scalar spectral theorem and eigenvalues

As promised in the introduction, we review some well known results on eigenvalues of non-singular transformations and spectral measures of measure preserving transformations. Firstly the

SCALAR SPECTRAL THEOREM. *Let $S \in G_m$, then there exists a finite positive measure $\tilde{\sigma}_S$ on $[0,1)$ and a continuous, symmetric sesquilinear map*

$$h_S : L^2(m) \times L^2(m) \to L^1(\tilde{\sigma}_S).$$

such that

(a) $\int_X f\bar{g} \cdot S^n dm = \int_0^1 e^{2\pi i n s} h_S(f,g)(s) d\tilde{\sigma}_S(s)$ *for every* $n \in \mathbf{Z}$, $f, g \in L^2(m)$,

(b) $h_S(f,f) \geqq 0$ *for every* $f \in L^2(m)$ *and for every* $h \in L^1(\tilde{\sigma}_S)$, $h \geqq 0$ *there is a* $g \in L^2(m)$ *with* $h_S(g,g) = h$.

This result, in which $f \to f \cdot S$ may be replaced by any unitary operator on the separable Hilbert space $L^2(m)$, is proved using G. Herglotz's theorem characterising positive definite sequences to show that

$$\int_X f\bar{f} \cdot S^n dm = \int_0^1 e^{2\pi i n s} d\mu_f(s)$$

for some positive measure μ_f on $[0,1)$, and then using standard (separable) Hilbert space techniques to "put all the μ_f's together".

Now, since $\mu_{1_X} = \delta_0$ we always have that $\tilde{\sigma}_S(\{0\}) > 0$. The measure $\sigma_S = \tilde{\sigma}_S - \tilde{\sigma}_S(\{0\})\delta_0$ is called the scalar spectral measure of S, and S's spectral type is the measure class of σ_S (that is $\{\mu : \mu \sim \sigma_S\}$).

Secondly, the

EIGENVALUE THEOREM. *Suppose* $T \in G^e$. *Let* $e(T) = \{s \in [0,1) : f : X \to \mathbf{T}$ *measurable with* $f \cdot T = e^{2\pi i s} f \bmod m\}$. *Then* $e(T)$ *is a Borel subgroup of* $[0,1)$ *and there is a jointly measureable function* (*Lebesgue* $\times$ *Borel*) $\eta : X \times e(T) \to \mathbf{T}$ *such that* $\eta(Tx, s) = e^{2\pi i s}\eta(x,s)$ *for every* $s \in e(T)$, $x \in X_s$ *where* $m(X_s^c) = 0$.

An indication of how to prove this is given in [2].

It is now standard to prove the

MULTIPLIER THEOREM. *If* $S \in G_m^e$ *and* $T \in G^e$ *then* $S \times T$ *is ergodic if and only if* $\sigma_S(e(T)) = 0$.

Since any bounded invariant function for $S \times T$ defines a measurable map $F : X \to L^2(m)$ with $F(Tx) \cdot S = F(x)$, whence, for every $g \in L^2(m)$,

$$h_S(F(Tx), g)(s) = e^{2\pi i s} h_S(F(x), g)(s), \qquad \sigma_S\text{-a.e.}$$

and, if $\mu_f(e(T)) = 1$ ($f \in L^2(m)$), one can use the function η in the eigenvalue theorem, and standard techniques to get $F : X \to \overline{\mathrm{sp}}\{f \cdot S^n\}_{n \in \mathbf{Z}}$ with $F(Tx) \cdot S = F(x)$. (Here, $\overline{\mathrm{sp}}\{f \cdot S^n\}_{n \in \mathbf{Z}}$ denotes the closed linear span of $\{f \cdot S^n\}_{n \in \mathbf{Z}}$ in L^2.)

§3. Towers over the adding machine

In this section we recall from [1], [2] some results on dyadic towers over the adding machine, whose eigenvalue groups and recurrence properties are controllable.

Let $\Omega = \{0,1\}^{\mathbf{N}}$, $\mathcal{A}$ be the σ-field generated by cylinder sets, and $P = (\frac{1}{2}, \frac{1}{2})^{\mathbf{N}}$ — Bernoulli measure.

Suppose $x = (\xi_1(x), \xi_2(x), \ldots) \in \Omega$; let $l(x) = \inf\{n \geqq 1 : \xi_n(x) = 0\}$, then $x = (1, \ldots, 1, 0, \xi_{l+1}, \xi_{l+2}, \ldots)$.

The adding machine is defined by $\tau x = (0, \ldots, 0, 1, \xi_{l+1}, \ldots)$. It is easy to see that $(\Omega, \mathcal{A}, P, \tau)$ is an ergodic measure preserving transformation.

Let $\gamma(n) \in \mathbf{N}$ for $n \geqq 1$. The dyadic height function with heights $\gamma(n)$ is $\phi(x) = \gamma(l(x))$ and the dyadic tower over the adding machine with height function ϕ is defined on

$$Y = \{(x,n) : x \in \Omega, \mathbf{N} \ni n \leqq \phi(x)\},$$

$$\mathcal{C} = \bigvee_{n=1}^{\infty} (\mathcal{A} \cap [\phi \geqq n], n), \mu = \sum_{n=1}^{\infty} P_{(\mathcal{A} \cap [\phi \geqq n], n)},$$

by

$$T(x,n) = \begin{cases} (x, n+1), & \phi(x) \geqq n+1, \\ (\tau x, 1), & \phi(x) = n. \end{cases}$$

Then ([7]) $(Y, \mathscr{C}, \mu, T)$ is a conservative, ergodic, measure-preserving transformation, and ([6]), $\mu(Y) = \int_\Omega \phi dP$. Since $(Y, \mathscr{C}, \mu)$ is separable and σ-finite, we may consider T as an element of G^e.

The towers over the adding machine used in this article are constructed by: Choosing $K \subseteq \mathbf{N}$, $K = \{k(\nu)\}_{\nu=0}^\infty$, where $k(\nu) < k(\nu+1)$, and setting $\gamma(1) = 2^{k(0)}$ and, for $n \geqq 2$,

$$\gamma(n) = 2^{k(n-1)} - \sum_{\nu=0}^{n-2} 2^{k(\nu)} \in \mathbf{N}.$$

If $(Y, \mathscr{C}, \mu, T)$ is constructed from $K \subseteq \mathbf{N}$ then

T is boundedly rationally ergodic with asymptotic type equivalent to $2^{|K \cap [1, \log_2 n]|}$ and, in particular,

$$\varliminf_{n\to\infty} \sum_{k=0}^{n-1} \mu(A \cap T^{-k}B)/2^{|K\cap[1,\log_2 n]|} > 0$$

for every $A, B \in \mathscr{C}$, $\mu(A), \mu(B) > 0$. (See [1].)

If $s \in [0,1)$ and $\sum_{n=1}^\infty |1 - e^{2\pi i 2^{k(n)}s}| < \infty$ then $s \in e(T)$. (See [1], [3].)

If $s \in e(T)$ then $e^{2\pi i 2^{k(n)}s} \to_{n\to\infty} 1$. (See [2].)

§4. In general, a transformation has small spectrum

We begin with the

MAIN LEMMA. *Suppose that $n_k, m_k \in \mathbf{N}$ and that $n_{k+1} > n_k + m_k + k$. For $L \subseteq \mathbf{N}$, $|L| = \infty$, let $K(L) = \bigcup_{k\in L}[n_k, n_k + m_k] \cap \mathbf{N}$, and T_L denote the dyadic tower over the adding machine constructed in §3 using $K(L)$, then $E(\{T_L : L \subseteq \mathbf{N}, |L| = \infty\})$ is meagre in G_m.*

PROOF. Recall that X is the unit interval $[0,1)$. Let $\mathscr{D}$ denote the dyadic sets in X, that is, all finite unions of dyadic intervals. Then $\mathscr{D}$ is a countable algebra, and m-dense in $\mathscr{B}$. A metric for the weak topology on G_m is given by

$$d(S_1, S_2) = \sum_{n=1}^\infty \frac{1}{2^n}(m(S_1 D_n \triangle S_2 D_n) + m(S_1^{-1} D_n \triangle S_2^{-1} D_n))$$

where $\mathscr{D} = \{D_n\}_{n=1}^\infty$.

Thus, the set

$$Z = \bigcap_{q=1}^\infty \bigcup_{k=q}^\infty \bigcap_{\nu=1}^q \left\{ S \in G_m : \sum_{j=n_k}^{n_k+m_k} m(D_\nu \triangle S^{-2^j} D_\nu)^{1/2} < 1/2^{q/2} \right\}$$

is a G_δ set in G_m.

Suppose that $S \in Z$, then $\exists k(q) \to \infty$ such that

$$\sum_{j=n_{k(q)}}^{n_{k(q)}+m_{k(q)}} m(D_\nu \triangle S^{-2} D_\nu)^{\frac{1}{2}} < 1/2^{q/2}$$

for $1 \leqq \nu \leqq q$, which implies, if $L = \{k(q)\}_{q=1}^{\infty}$, that

$$\sum_{j \in K(L)} m(D \triangle S^{-2^j} D)^{\frac{1}{2}} < \infty \qquad \text{for every } D \in \mathcal{D}.$$

Now

$$\begin{aligned} m(D \triangle S^{-2^j} D) &= \int_X |1_D - 1_D \cdot S^{2^j}|^2 dm \\ &= 2\left(m(D) - \int_x 1_D 1_D \cdot S^{2^j} dm\right) \\ &= 2\left(m(D) - \int_0^1 e^{2\pi i s 2^j} h(1_D, 1_D) \cdot (s) d\tilde{\sigma}_S(s)\right) \\ &= 2\int_0^1 (1 - \cos 2\pi 2^j s) h(1_D, 1_D)(s) d\tilde{\sigma}_S(s) \\ &= \int_0^1 |1 - e^{2\pi i 2^j s}|^2 h(1_D, 1_D)(s) d\tilde{\sigma}_S(s) \\ &\geqq \frac{1}{m(D)} \left(\int_0^1 |1 - e^{2\pi i 2^j s}| h(1_D, 1_D)(s) d\tilde{\sigma}_S(s)\right)^2. \end{aligned}$$

Hence, for every $s \in Z$ there exists $L \subseteq \mathbf{N}$, $|L| = \infty$ so that for every $D \in \mathcal{D}$:

$$\int_0^1 \sum_{j \in K(L)} |1 - e^{2\pi i 2^j s}| h(1_D, 1_D)(s) d\tilde{\sigma}_S(s) < \infty$$

which implies that $\sigma_S(e(T_L)) > 0$ (in fact, it implies that $\sigma_S(e(T_L)^c) = 0$), which in turn entails the non-ergodicity of $S \times T_L$. Thus

$$Z \subseteq G_m - E(\{T_L : L \subseteq \mathbf{N}, |L| = \infty\}).$$

To complete the proof of the main lemma, we show that Z is dense, and hence residual, in G_m. To do this we find an ergodic $S_0 \in Z$ with sufficiently many conjugates in Z.

From the condition $n_{k+1} > m_k + k$ it follows that there is an irrational $\alpha \in [0,1)$ with $\alpha = \sum_{n=1}^{\infty} \varepsilon_n / 2^n$ where $\varepsilon_n = 0, 1$ for $n \geqq 1$ and $\varepsilon_j = 0$ for $j = n_k + \nu$ $(0 \leqq \nu \leqq m_k + k,\ k \geqq 1)$. This means that

$$((2^{n_k + \nu} \alpha)) \leqq \frac{1}{2^{m_k - \nu + k}} \qquad (0 \leqq \nu \leqq m_k).$$

(Here, $((x))$ denotes the fractional part of x.)

Let $S_\alpha(x) = ((x+\alpha))$. Then $S_\alpha \in G_m \cap G^e$.

Note first that if $I \subseteq [0,1)$ is an interval, then $m(I \triangle S^{-n} I) \leqq 2\langle((n\alpha))\rangle$ where $\langle x \rangle = x \wedge (1-x)$.

Now, suppose that $A = \bigcup_{j=1}^K I_j$ where I_j are intervals. Then, since $A \triangle S_\alpha^{-n} A \subseteq \bigcup_{j=1}^K I_j \triangle S_\alpha^{-n} I_j$, we have that $m(A \triangle S_\alpha^{-n} A) \leqq 2K\langle((n\alpha))\rangle$.

For $D \in \mathcal{D}$ a dyadic set, let $N(D)$ denote the minimal number of component intervals in the union forming D. Then

$$m(D \triangle S_\alpha^{-n} D) \leqq 2N(D)\langle((n\alpha))\rangle \quad \text{and} \quad m(D \triangle S^{-2^{n_k+\nu}} D) \leqq 2N(D)/2^{m_k-\nu+k}$$

$$\text{for } 0 \leqq \nu \leqq m_k,\ k \geqq 1,$$

whence

$$\sum_{\nu=n_k}^{n_k+m_k} m(D \triangle S_\alpha^{-2^\nu} D)^{\frac{1}{2}} \leqq \frac{2\sqrt{N(D)}}{\sqrt{2}-1} \cdot \frac{1}{2^{k/2}}.$$

Let $G_m^0 = \{\Pi \in G_m : \Pi\mathcal{D} = \mathcal{D}\}$. We show that $\Pi^{-1} S_\alpha \Pi \in Z$ for every $\Pi \in G_m^0$. Choose $\Pi \in G_m^0$. Let $S_\pi = \Pi^{-1} S_\alpha \Pi$, then for $n \geqq 1$

$$m(D \triangle S_\pi^{-n} D) = m(D \triangle \Pi^{-1} S_\alpha^{-n} \Pi D) = m(\Pi D \triangle S_\alpha^{-n} \Pi D).$$

If $D \in \mathcal{D}$ then so is ΠD. Let $q \geqq 1$ and let

$$N_q = \max_{1 \leqq \nu \leqq q} N(\Pi D_\nu).$$

Choose $k \geqq q$ so that

$$\frac{2\sqrt{N_q}}{\sqrt{2}-1} \frac{1}{2^{k/2}} < \frac{1}{2^{q/2}}.$$

Then

$$\sum_{j=n_k}^{n_k+m_k} m(D_\nu \triangle S_\pi^{-2^j} D_\nu)^{\frac{1}{2}} = \sum_{j=n_k}^{m_k+m_k} m(\Pi D_\nu \triangle S_\alpha^{-2^j} \Pi D_\nu)^{\frac{1}{2}} \leqq \frac{2\sqrt{N_q}}{\sqrt{2}-1} \frac{1}{2^{k/2}} < \frac{1}{2^{q/2}}$$

and $S_\Pi \in Z$.

Since S_α is ergodic, it follows from the proof of the conjugacy lemma in [5] that $\{S_\pi : \Pi \in G_m^0\}$ is dense in G_m, whence Z is dense, and the main lemma is proven. □

THEOREM 1. *If $\rho(t) \downarrow 0$ as $t \downarrow 0$ then $E(G^e(\rho))$ is meagre in G_m.*

PROOF. We begin by constructing $n_k, m_k \in \mathbf{N}$ with $n_{k+1} < n_k + m_k + k$ so that $H_\rho(e(T_L)) = 0$ for every $L \subseteq N$, $|L| = \infty$. To obtain this, given n_k, choose m_k so large that

$$\rho(1/2^{n_k+m_k}) \leqq 1/2^{n_k+k} \quad \text{and} \quad n_{k+1} > n_k + m_k + k.$$

Now

$$e(T_L) \subseteq \left\{ s \in [0,1) : e^{2\pi i s 2^j} \underset{\substack{j\to\infty \\ j \in K(L)}}{\longrightarrow} 1 \right\} \subseteq \bigcup_{q=1}^{\infty} A_q$$

where $A_q = \{s \in [0,1) : \langle((2^{n_k+\nu}s))\rangle < \frac{1}{8} \text{ for } 0 \leqq \nu \leqq m_k,\ k \in L,\ k > q\}$.

It suffices to show that $H_\rho(A_q) = 0$ for every $q \geqq 1$. Note that

$$A_q \subseteq B_q = \left\{ s = \sum_{n=1}^{\infty} \varepsilon_n/2^n ;\ \varepsilon_n = 0,1 \text{ and } \varepsilon_{n_k+1} = \cdots = \varepsilon_{n_k+m_k} \text{ for } k \in L,\ k \geqq q \right\}.$$

Now suppose that $k \in L$ and $k \geqq q$. It is possible to cover B_q with dyadic intervals of the form

$$\left[\sum_{j=1}^{n_k+m_k} \varepsilon_j/2^j,\ \sum_{j=1}^{n_k+m_k} \varepsilon_j/2^j + 1/2^{n_k+m_k} \right)$$

where $\varepsilon_j = 0,1$ for every j and $\varepsilon_{n_k+1} = \varepsilon_{n_k+2} = \cdots = \varepsilon_{n_k+m_k}$. There are 2^{n_k+1} intervals like this, each one having length $1/2^{n_k+m_k}$ and so

$$\inf \left\{ \sum \rho(|I|) : B_q \subseteq \cup I,\ |I| \leqq 1/2^{nk} \right\} \leqq \rho(1/2^{n_k+m_k}) 2^{n_k+1} \leqq 1/2^k.$$

Hence ($|L| = \infty$) $H_\rho(B_q) = 0$ for every q, and $H_\rho(e(T_L)) = 0$ for every $L \subseteq \mathbf{N}$, $|L| = \infty$. Thus

$$\{T_L : L \subseteq \mathbf{N}, |L| = \infty\} \subseteq G^e(\rho) \quad \text{and} \quad E(G^e(\rho)) \subseteq E(\{T_L : L \subseteq \mathbf{N}\ |L| = \infty\})$$

which is meagre by the main lemma. □

THEOREM 2. *If* $c_n \downarrow$ *as* $n \uparrow \infty$, $\sum_{n=1}^{\infty} c_n = \infty$, *and* $c = (c_1, c_2, c_3, \ldots)$, *then* $E(G^e(c))$ *is meagre.*

PROOF. Again, we use the main lemma, constructing n_k, m_k so that T_L is c_n-recurrent for every $L \subseteq \mathbf{N}$, $|L| = \infty$.

Since $\sum_{k=1}^{\infty} c_k = \infty$ and $c_n \downarrow$ as $n \uparrow$ we have that $\sum_{n=1}^{\infty} 2^n c_{2^n} = \infty$. Given n_k, choose m_k so large that

$$\sum_{j=n_k+1}^{n_k+m_k} 2^j c_{2^j} \geqq 2^{n_k}$$

and then choose $n_{k+1} > n_k + m_k + k$.

Now, suppose that $L \subseteq \mathbf{N}$, $|L| = \infty$. One of the results stated in §3 was that $T_L \in G^e(\{a_L(n)\})$, where

$$a_L(n) = 2^{|K(L)\cap[1,\log_2 n]|}$$

from which it will follow that T_L is c_n-recurrent if $\sum_{n=1}^{\infty}(c_n - c_{n+1})a_L(n) = \infty$.

A manipulation shows that this series diverges with

$$\sum_{n=1}^{\infty} 1_{K(L)}(n)c_{2^n}2^{|K(L)\cap[1,n]|} \geqq \sum_{k\in L} 2^{-n_k} \sum_{j=n_k+1}^{m_k+m_k} 2^j c_{2^j}$$

and this latter series diverges by the construction of n_k and m_k. Thus $\{T_L : L \subseteq \mathbf{N}, |L| = \infty\} \subseteq G^e(c)$ and hence

$$E(G^e(c)) \subseteq E(\{T_L : L \subseteq \mathbf{N} \mid |L| = \infty\})$$

which is meagre by the main lemma. □

§5. A residual multiplier property

THEOREM 3. *For every $a(n)\to\infty$, $a(n)/n\to 0$, $E(G^e(\{a(n)\}))$ is residual.*

We need:

LEMMA 4. *Suppose that $T \in G^e(\{a(n)\})$ and $S \in G_m$ are such that $S \times T$ is not ergodic. Let $\alpha(n)\to_{n\to\infty}0$, then there exists $A \in \mathcal{B}$, $0 < m(A) < 1$ and $K \subseteq \mathbf{N}$ such that*

$$|K \cap [1,n]|/\alpha(n)a(n)\to\infty \quad \text{and} \quad m(A \,\triangle\, S^{-n}A) \underset{\substack{n\to\infty \\ n\in K}}{\longrightarrow} 0.$$

PROOF. Since $S \times T$ is not ergodic there is a set $\tilde{A} \in \mathcal{B}\otimes\mathcal{B}$ such that $0 < m\otimes m(\tilde{A}) < 1$ and $1_{\tilde{A}}(Sy, Tx) = 1_{\tilde{A}}(y,x)$ for $m\times m$-a.e. $(y,x) \in X\times X$, Define $F: X\to L^2(m)$ by $F(x)(y) = 1_A(y,x)$. Then $F(Tx) = F(x)\cdot S^{-1}$.

For $f \in L^2(m)$ let $A(f,\varepsilon) = \{x \in X : \|F(x) - f\|_2 < \varepsilon\}$. It follows easily from the separability of $L^2(m)$ that $m(A(F(x),\varepsilon) > 0$ $\forall\varepsilon > 0$ a.e. Thus there is an $x_0 \in X$ such that $m(A(F(x_0),\varepsilon)) > 0$ for every $\varepsilon > 0$ and such that $0 < \int_X F(x_0)dm < 1$ (m-a.e. $x_0 \in X$ will do).

Now $F(x_0) = 1_{A_0}$ where $A_0 \in \mathcal{B}$, $0 < m(A_0) < 1$. Suppose $x \in A(F(x_0),\varepsilon)$, $n \geqq 1$ and $T^n x \in A(F(x_0),\varepsilon)$, then

$$\begin{aligned} m(A_0 \triangle S^{-n}A_0)^{\frac{1}{2}} &= \|F(x_0) - F(x_0)\cdot S^n\|_2 \\ &< \|F(x_0) - F(x)\|_2 + \|F(x) - F(x_0)\cdot S^n\|_2 \\ &= \|F(x_0) - F(x)\|_2 + \|F(x)\cdot S^{-n} - F(x_0)\|_2 \\ &= \|F(x_0) - F(x)\|_2 + \|F(T^n x) - F(x_0)\|_2 \\ &< 2\varepsilon. \end{aligned}$$

Thus setting $K(\varepsilon)=\{n\geqq 1: m(A_0\triangle S^{-n}A_0)<\varepsilon\}$ we have that

$$1_{A(F(x_0),\frac{1}{2}\vee\varepsilon)}(x)\sum_{k=1}^{n}1_{A(F(x_0),\frac{1}{2}\vee\varepsilon)}(T^kx)\leqq|K(\varepsilon)\cap[1,n]|.$$

Integrating this inequality on X and noting that $T\in G^e(\{a(n)\})$, we see that for every $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ so that $|K(\varepsilon)\cap[1,n]|>\delta(\varepsilon)a(n)$ for $\varepsilon>0$ and $n\geqq 1$. Choose $\varepsilon_k\to 0$ and $n_k<n_{k+1}\uparrow\infty$ so that $\sqrt{\alpha(n)}<\delta(\varepsilon_k)$ for every $n>n_k$. Setting

$$K=\bigcup_{k=1}^{\infty}[n_k+1,n_{k+1}]\quad K(\varepsilon_k),$$

we obtain that

$$|K\cap[1,n]|>\sqrt{\alpha(n)}a(n)\quad\text{for } n>1\quad\text{and}\quad m(A_0\triangle S^{-n}A_0)\underset{\substack{n\to\infty\\ n\in K}}{\longrightarrow}0.\qquad\square$$

The proof of Lemma 4 is similar to that of lemma 4 in [1]. One can adapt that proof to show that if $T\in G^e(\{a(n)\})$ has a σ-finite, infinite invariant measure, then K can be chosen with $|K\cap[1,n]|/a(n)\to\infty$.

Proof of Theorem 3. Suppose $a(n)\to\infty$, $a(n)/n\to 0$. Suppose that $\alpha(n)\to 0$, but $a_1(n)=\alpha(n)a(n)\to\infty$.

Let $\mathcal{D}=\{D_\nu\}_{\nu=1}^{\infty}$ be the dyadic sets and set

$$Z=\bigcap_{q=1}^{\infty}\bigcup_{n=q}^{\infty}\bigcup_{\substack{L\subseteq[1,n]\\|L|\geqq n-a_1(n)/q}}\bigcap_{\mu,\nu=1}^{q}\bigcap_{k\in L}\left\{S\in G_m:|m(D_\nu\cap S^{-k}D_\mu)-m(D_\nu)m(D_\mu)|<\frac{1}{q}\right\}.$$

The set Z is clearly a G_δ set in G_m. We prove Theorem 3 by showing that Z is dense and $Z\subseteq E(G^e(\{a(n)\}))$.

Suppose $S\in Z$, and for $q>1$, let

$$L_q=\left\{n\geqq 1:|m(D_\mu\cap S^{-n}D_\nu)-m(D_\mu)m(D_\nu)|<\frac{1}{q}\text{ for }1\leqq\mu,\nu\leqq q\right\}.$$

The condition $S\in Z$ entails the existence of a sequence $n(q)\uparrow\infty$ such that $|L_q\cap[1,n(q)]|>n(q)-a_1(n(q))/q$. Set

$$L=\bigcup_{q=1}^{\infty}[n(q)+1,n(q+1)]\cap L_q.$$

Since $L_{q+1}\subseteq L_q$ we have that for $q\geqq 1$, $|L\cap[1,n(q)]|\geqq n(q)-a_1(n(q))/q$ and hence

$$\varliminf_{n\to\infty}|L^c\cap[1,n]|/a_1(n)=0.$$

From the definition of L, we have that for any $D, D' \in \mathcal{D}$

$$m(D \cap S^{-n}D) \underset{\substack{n\to\infty \\ n \in L}}{\longrightarrow} m(D)m(D').$$

Since $\mathcal{D}$ generates $\mathcal{B}$, this last is true for any $D, D' \in \mathcal{B}$. On the other hand it is evident that any S with the above property is in Z. So $S \in Z$ if and only if there is a set $L \subseteq \mathbf{N}$, such that

$$\varliminf_{n\to\infty} |L^c \cap [1,n]|/a_1(n) = 0 \quad \text{and} \quad m(A \cap S^{-n}B) \underset{\substack{n\to\infty \\ n \in L}}{\longrightarrow} m(A)m(B)$$

for every $A, B \in \mathcal{B}$.

Now, Z is patently dense in G_m as it contains all mixing transformations. Hence Z is residual.

Now, suppose that $S \in Z$, $T \in G^e(\{a(n)\})$ and $S \times T$ is not ergodic. By Lemma 4 there is an $A_0 \in \mathcal{B}$, $0 < m(A_0) < 1$ and a $K \subseteq [1, n]$ with

$$m(A_0 \triangle S^{-n}A_0) \underset{\substack{n\to\infty \\ n \in K}}{\longrightarrow} 0 \qquad \left(\text{whence } m(A_0 \cap S^{-n}A_0) \underset{\substack{n\to\infty \\ n \in K}}{\longrightarrow} m(A_0)\right)$$

and $|K \cap [1, n]|/a_1(n) \to \infty$.

But since $S \in Z$ there is an L and $n(q) \to \infty$ so that

$$m(A_0 \cap S^{-n}A_0) \underset{\substack{n\to\infty \\ n \to L}}{\longrightarrow} m(A_0)^2 \quad \text{and} \quad |L \cap [1, n(q)]| > n(q) - a_1(n(q))/q.$$

Since $m(A_0)^2 < m(A_0)$, we must have that for some N, $K \cap [N, \infty] \subseteq L^c \cap [N, \infty]$ whence

$$|K \cap [1, n(q)]| < |L^c \cap [1, n(q)]| + N < a_1(n(q))/q + N,$$

contradicting $|K \cap [1, n]|/a_1(n) \to \infty$. Thus $Z \subseteq E(G^e(\{a(n)\}))$ and the latter set is therefore residual. □

References

1. J. Aaronson, *Rational ergodicity, bounded rational ergodicity ad some continuous measures on the circle*, Isr. J. Math. **33** (1979), 181–197.

2. J. Aaronson, *The eigenvalues of non-singular transformations*, Isr. J. Math. **45** (1983), 297–312.

3. H. Furstenberg and B. Weisss, *The finite multipliers of infinite ergodic transformations*, Springer Lecture Notes, Vol. 668, Springer-Verlag, Berlin, 1978, pp. 127–132.

4. P. Halmos, *In general a transformation is mixing*, Ann. Math. **45** (1944), 776–782.

5. P. Halmos, *Lectures on Ergodic Theory*, Chelsea, 1955.

6. M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Am. Math. Soc. **53** (1947), 1002–1010.

7. S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Sci. Tokyo **19** (1943), 635–641.

8. U. Krengel, *Classification of states for operators*, Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability, 1966.